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ORIGINAL PAPER

Tutte and Jones polynomials of links, polyominoes and graphical recombination patterns

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Abstract In order to facilitate recognition of polymer graphs and patterns obtained by graphical recombination, we analyze polynomial invariants, graphs and knots associated to polyominoes, polyiamonds, and polyhexes.

Keywords Knot · Link · Tait graph · Middle graph · Polyiamond · Polyomino · Polyhex · Tutte polynomial · Jones polynomial · Polymer

1 Introduction

Knot theory basics and Conway notation of knots and links (or shortly *KLs*) used in this paper follows the works [1–4]. A *graph* is a pair (V, E), where V denotes the *set of vertices* and $E \subseteq V \times V$ the *edge set*. We consider only *undirected* graphs, i.e. edge $e \in E$ can be denoted by (A, B) or (B, A), where $A, B \in V$ denote its endpoints. A *loop* is an edge (x, x) connecting a vertex to itself, and a bridge is an edge whose removal disconnects two or more vertices (i.e. there is no other path between them) [5,6]. Tutte polynomial can be defined recursively using two operations on graphs, *edge deletion* denoted by $G - e$, and *edge contraction* G/e (see Fig. 1).

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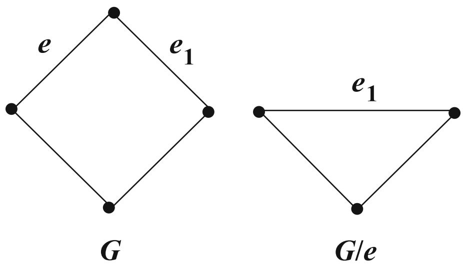

Fig. 1 Graph G and a graph G/e obtained by contracting edge e

Definition 1 The *Tutte polynomial* of a graph $G(V, E)$ is a two-variable polynomial defined as follows:

$$T(G) = \begin{cases} 1 & E(\emptyset) & (1) \\ xT(G/e) & e \in E \text{ if } e \text{ is a bridge} & (2) \\ yT(G-e) & e \in E \text{ if } e \text{ is a loop} & (3) \\ T(G-e) + T(G/e) & e \in E \text{ if } e \text{ is neither a loop nor a bridge} & (4) \end{cases}$$

The definition of a Tutte polynomial outlines a simple recursive procedure for computing it, but the order in which rules are applied is not fixed.

According to Thistlethwaite's Theorem, the Jones polynomial of an alternating link, up to a factor, can be obtained from Tutte polynomial by replacements: $x \to -x$ and $y \to -\frac{1}{x}$ [7–10]. HOMFLYPT and Alexander polynomials of Jaeger links [11] can be computed directly from the Tutte polynomials of the corresponding graphs.

Lin [12], Chang and Shrock [10], Wu and Wang [13], and Jin and Zhang [14] have analyzed zeros of the Jones polynomial. Moreover, Champanarekar and Kofman [15] explained numerous experimental observations by different authors about the distribution of roots of Jones polynomials for various families of knots and links. In [16] we provided plots of zeroes of Jones polynomials, called "portraits of families", for all families of knots and links derived from source links with at most $n = 10$ crossings.

The Tutte polynomial is the same as the partition function for the Potts model and physicists have done many calculations of Potts models [17].

A *cut-vertex* (or articulation vertex) of a connected graph is a vertex whose removal disconnects the graph [18]. In general, a cut-vertex is a vertex of a graph whose removal increases the number of components [19]. A *block* is a maximal biconnected subgraph of a given graph.

One-point union or *block sum* of two (disjoint) graphs G_1 and G_2, neither of which is a vertex graph, which we shall denote as $G_1 * G_2$, is of the particular interest. This one-point union is such that the intersection of G_1 and G_2 can only consist of a vertex [20].

Decomposition of a graph G into a finite number of blocks G_1,...,G_n, denoted by

$$G = G_1 * G_2 * \ldots G_n$$

is called the *block sum* of G_1,...,G_n. The following formula holds for the Tutte polynomial of the block sum:

$$T(G_1 * G_2 * \dots G_n) = T(G_1)T(G_2)\dots T(G_n).$$

The Tutte polynomial is invariant with regard to *block-moves*, where a block can be moved along the edges of the remaining part of the graph.

A *dual graph* $\overline{G}$ of a given planar graph G is a graph which has a vertex for each plane region of G, and an edge for each edge in G joining two neighboring regions, for a certain embedding of G. The Tutte polynomial of $\overline{G}$ can be obtained from $T(G)$ by replacements $x \to y$, $y \to x$, i.e. $T(\overline{G})(x, y) = T(G)(y, x)$.

There is a nice bijective correspondence between *KL*s and (signed, planar) graphs: to obtain a graph from a *KL* diagram, first color every other region of the *KL* diagram black or white, so that the infinite outermost region is black. In the *checker-board coloring* (or *Tait coloring*) of the plane obtained, put a vertex at the center of each white region. Two vertices of a graph are connected by an edge if there was a crossing between corresponding regions in a *KL* diagram. In addition, we can label each edge of a graph by the sign of its corresponding vertex of the *KL* diagram. Family of graphs corresponding to a family of link diagrams L will be denoted by $G(L)$.

The reverse construction turns a signed planar graph G into a knot or link. The process starts by placing a 4-valent vertex in the middle of each edge of the graph G and connecting the vertices belonging to adjacent edges of G in such a way that no additional vertices are obtained. The obtained graph is called *middle graph*. Finally, we use the sign information to turn each vertex into either positive or negative crossing.

The idea of studying graphs via knots and links associated to them is not new—it has been applied in the study of protein structures. In this case, we restrict our attention to unsigned planar graphs, hence we need to consider only alternating *KL* diagrams. Moreover, sometimes it is more convenient to use a slightly different way of associating knots to graphs, by introducing a digon, instead of a crossing, in the middle of each edge. Based on this idea, Chen, Dill [21] and Emmert-Steib [22] proposed using knot polynomials for distinguishing protein structures. More precisely, they were able to algorithmically compute HOMFLYPT and Alexander polynomials of *KL*s associated to graphs of protein structures, via the Tutte polynomial. One of the characteristic reduced protein graphs corresponding to the antiparallel sheet-confirmation is the simplest n-polyomino, whose contact graph is a rectilinear chain. Also, it is well known that the structure of bezenoids is described by polyhexes [23].

In this paper we analyze different knot polynomials of alternating *KL*s obtained from middle graphs corresponding to n-iamonds, n-minoes, and n-hexes. This method can be applied to connected parts of plane tilings and used for the recognition of graphical recombinations. In the last section we give general formulae for the Tutte polynomials of links corresponding to different "families" of polyiamonds, polyominoes and polyhexes. We extend these results to graphs corresponding to antiparallel sheet derived from polyiamionds and polyhexes, and parallel sheet for polyamonds and polyhexes. The method developed in this paper can be applied to antiparallel and helix graphs, and possibly to graphs corresponding to more complex and interesting protein structures.

Based on the formulas for the Tutte polynomial, we computed the corresponding Jones polynomials and provided plots of their zeroes (“portraits of families”) for different *KL* families corresponding to polyominoes (polyiamonds, polyhexes). All computations we done by *Mathematica* based program *LinKnot* [4] with some additional functions.

2 *KL*s derived from polyominoes

Numbers of different free n-minoes are given by the sequence A000105 from N. Sloane’s “The On-Line Encyclopedia of Integer Sequences.” For $1 \leq n \leq 15$ they are 1, 1, 2, 5, 12, 35, 108, 369, 1285, 4655, 17073, 63600, 238591, 901971, 3426576, respectively. Polyominoes, polyiamonds, and polyhexes are generated using the program “Polyominoes and Related Families” by Rangel-Mondragón [24]. Table 1 contains the number of different free polyominoes and different alternating *KL*s obtained as middle graphs from polyomino graphs for $1 \leq n \leq 8$. In order to recognize equal *KL*s we computed Tutte polynomials, Kauffman polynomials and Jones polynomials (for $n = 8$).

The n-minoes are grouped together (see a–j in Fig. 2), based on their corresponding *KL*s: the link 4 (4^2_1) corresponds to the group (a), knot 3 1 3 (7_4) to

Table 1 The number of different free polyominoes and different alternating *KL*s obtained from middle graphs of polyomino graphs for $1 \leq n \leq 8$

No. of polyominoes	1	1	2	5	12	35	108	369
No. of *KL*s	1	1	1	3	4	10	20	55

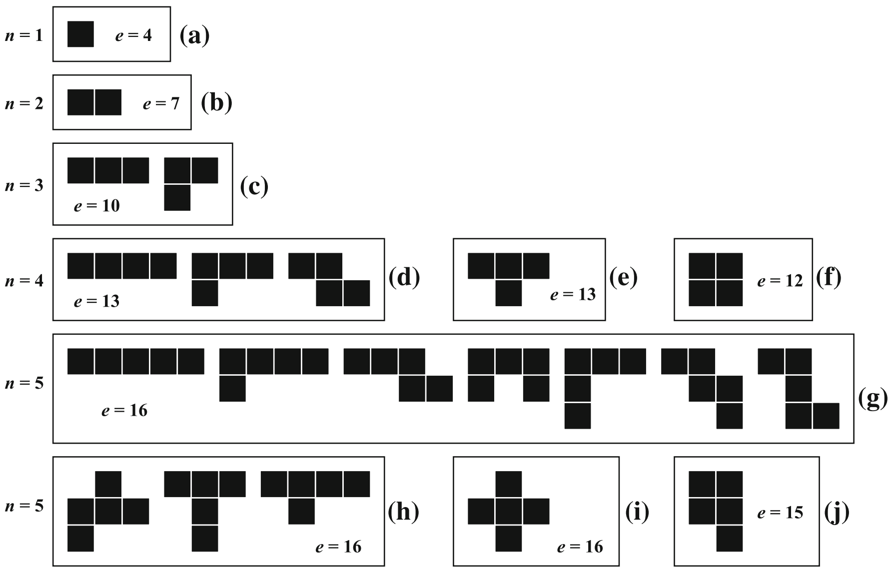

Fig. 2 Groups of polyominoes giving the same knot or link

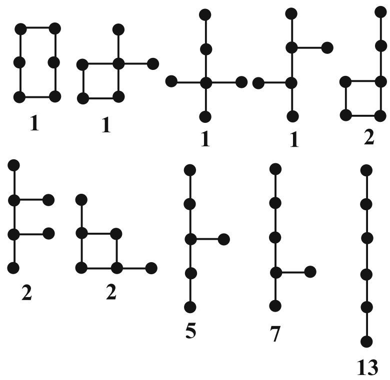

Fig. 3 Contact graphs and the numbers of their occurrence for 6-ominoes

(b), 2-component link 3 1 2 1 3 to (c) , knot 3 1 2 1 2 1 3 to (d), 3-component link 3 1, 3 1, 3 1+ to (e), 3-component link 8*2 : 2 : 2 : 2 to (f), and 2-component links 3 1 2 1 2 1 2 1 3, 3 1 2 1, 3 1, 3 1+, 3 1, 3 1, 3 1, 3 1, and 8*3 2 1 : 2 : 2 : 2, correspond to (g), (h), (i), and (j), respectively. The number of crossings of these links is equal to the number of edges e of the polyomino graph.

If we substitute each internal face of an n-mino (n-iamond, or n-hexe) P with a vertex and join vertices belonging to adjacent faces by an edge, we obtain a new graph with n vertices, called *contact graph*[1] $G'(P)$. Two different non-hollow n-minoes P_1 and P_2 give the same alternating knot or link *iff* their corresponding contact graphs $G'(P_1)$ and $G'(P_2)$ are isomorphic. A polyomino (polyiamond, polyhex) graph in which edges between adjacent faces are replaced by double edges is called *extended polyiomino (polyiamond, polyhex) graph*, respectively.

Figure 3 contains 6-omino contact graphs, together with the number of corresponding 6-ominoes among total of 35 different ones.

3 *KL*s derived from polyiamonds and polyhexes

Numbers of different free n-polyiamonds are given by the sequence A000577 from N. Sloane's "The On-Line Encyclopedia of Integer Sequences."

Table 2 contains the number of free polyiamonds and the number of different *KL*s derived from polyiamond graphs for $1 \leq n \leq 10$.

Numbers of different free n-polyhexes are given by the sequence A000228 from N. Sloane's "The On-Line Encyclopedia of Integer Sequences."

In Table 3 we give the number of free polyhexes and the number of different *KL*s derived from polyhex graphs for $1 \leq n \leq 8$.

In the same way as for non-hollow polyominoes, the number of non-hollow polyiamods and polyhexes is equal to the number of their mutually non-isomorphic contact graphs, respectively. It is well known that the Tutte polynomial is invariant with regard to block-moves, but in the case of polyominoes, it is the same for all non-hollow

[1] For a polyomino P, the graph $G'(P)$ is the subgraph of the dual graph of a polyomino graph, where connections with the external face of the polyomino are deleted.

Table 2 The number of free polyiamonds and the number of different *KL*s derived from polyiamond graphs for $1 \leq n \leq 10$

No. of polyiamonds	1	1	1	3	4	12	24	66	160	488
No. of *KL*s	1	1	1	2	2	5	7	15	27	60

Table 3 The number of free polyhexes and different *KL*s derived from polyhex graphs for $1 \leq n \leq 8$

No. of polyhexes	1	1	3	7	22	333	1, 448	6, 572
No. of *KL*s	1	1	2	4	8	54	156	457

polyominoes with isomorphic contact graphs. The same holds for non-hollow polyiamonds and polyhexes. The proof of this fact is very simple, because non-hollow polyomino (polyiamond, polyhex) graphs with isomorphic contact graphs have the same resolving trees for their Tutte polynomials. The same holds for extended polyomino (polyiamond, polyhex) graphs.

A hollow polynomino (polyiamond, polyhex) can have the contact graph isomorphic to the contact graph of some other polyomino (polyiamond, polyhex), but different Tutte polynomial (Fig. 5b). In this case, instead of contact graphs, we can consider complete dual graphs of polyomino graphs. Two polyominoes (polyiamonds, polyhexes) P_1 and P_2 have the same Tutte polynomials *iff* the duals of their graphs are isomorphic. For example, hollow polyiamond (Fig. 5a) and polyiamond (Fig. 5c) have the same contact graph, but different Tutte polynomials and non-isomorphic duals of their graphs. Hollow polyiamonds from Fig. 5a and b have the same Tutte polynomial, but their corresponding extended graphs have different Tutte polynomials.

4 Distinguishing graphs by *KL* polynomials

Free polyominoes (polyiamons, polyhexes) cannot be distinguished by *KL* polynomials (Tutte polynomial, Jones, HOMFLYPT, Alexander-Conway): these invariants detect only free polyominoes with different contact graphs. However, they can be efficiently distinguished by polynomials corresponding to *KL* diagrams, such as Liang polynomial, introduced by Liang and Jiang in 1982 [25] as a tool for recognizing amphicheirality of *KL* diagrams.

Given an oriented knot or link L, denote each oriented arc connecting two successive undercrossings by x_i ($i = 1, 2, \ldots, m$). Oriented arcs x_i are called *generators* of the link L.

Let us consider an oriented alternating *KL* diagram D with generators $g_1, \ldots, g_n$. In every vertex of D there are three generators: passing generator g_i, and incoming and outgoing generators g_j, g_k, respectively. The Liang polynomial $d_D(t) =$ is defined in the following way: if $\epsilon(V)$ is the sign of the crossing V, then $a_{ii} = t^{\epsilon(V)}$, $a_{ij} = s$ if the vertices i, j are connected with multiplicity s ($s = 0, 1, 2$), and $d_D(t) = \det(a_{ij})$.

Isomorphic *KL* diagrams have the same Liang polynomial. In order to increase the selectivity of this polynomial for links, we can use different variables for generators belonging to different components. However, ordinary Liang polynomial, computed

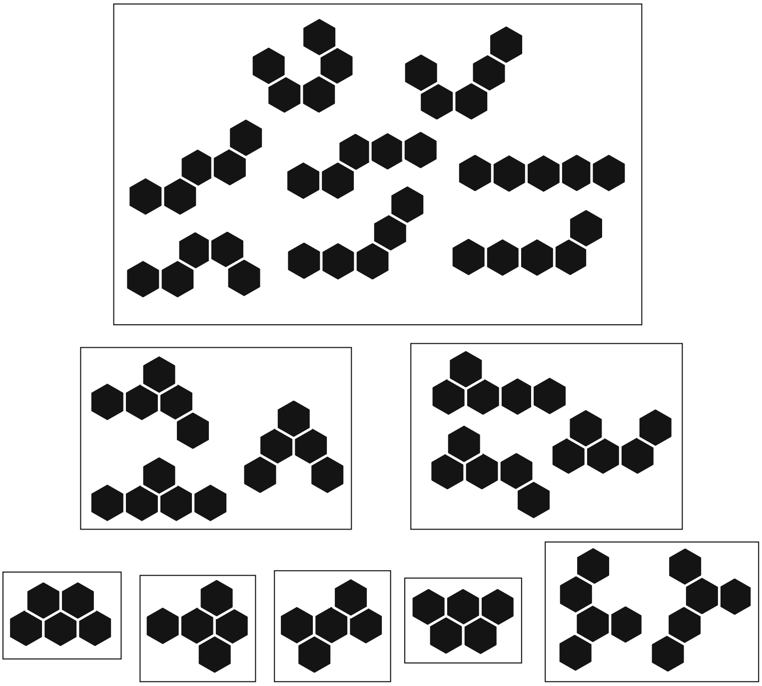

Fig. 4 Hexominoes with $n = 5$ grouped according their corresponding *KL*s, i.e., according the contact graphs

for *KL*s derived from polyominoes, polyiamonds, and polyhexes for $1 \leq n \leq 10$ was sufficient for distinguishing all free polyominoes, polyiamonds, and polyhexes. Since Liang polynomial computations are very fast, it is an efficient tool for recognizing graphs, based on their middle graphs (Fig. 4).

Probably the fastest way to recognize polyominoes, polyiamonds, or polyhexes is by using characteristic polynomials of their corresponding graphs. However, the characteristic polynomial, or the Liang polynomial are not the complete invariants of alternating knot diagrams, meaning that two non-isomorphic diagrams can give the same polynomial. Simpler invariant, minimal Dowker code of the link diagrams provides slow, but reliable solution.

In order to distinguish hollow polyominoes, polyiamonds, or polyhexes, we need additional data: the contact graph. For example, hollow polyiamonds (Fig. 5a, b) have isomorphic polyiamond graphs, but different contact graphs. In this case, we can consider extended polyomino (polyiamond, polyhex) graphs, with edges between adjacent faces doubled, and their corresponding characteristic polynomials. It is known that two nonisomorphic graphs may have the same characteristic polynomial, but in the case of polyiamonds characteristic polynomials computed for extended polyiamond graphs up to $n = 12$ distinguish all of them (including those corresponding to hollow polyiamonds).

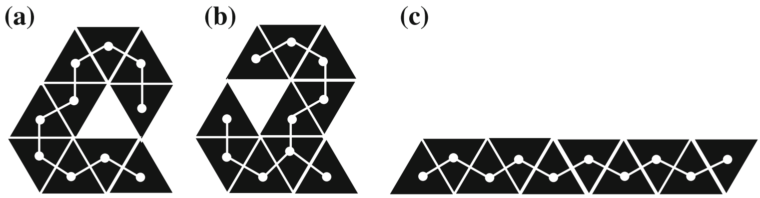

Fig. 5 Two hollow polyiamonds (**a**), **b** with isomorphic graphs and with different contact graphs, and polyiamond (**c**) which has the same contact graph as the hollow polyiamond (**a**)

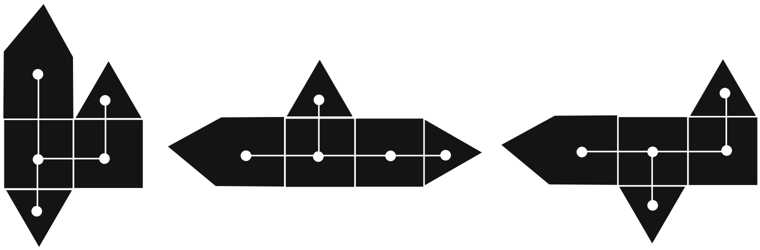

Fig. 6 Three shapes with the same Tutte polynomial

5 General case: graphical recombinations

In general, we can view any plane tiling as a graph, take its connected finite subgraph T and compute its Tutte and characteristic polynomial, or Tutte and characteristic polynomial of its extended graph. Characteristic polynomial will distinguish different geometric shapes T, and the Tutte polynomial will distinguish graphs with the same corresponding KLs. For example, shapes shown on Fig. 6 have different characteristic polynomials corresponding to their (extended) graphs, but the same Tutte polynomials, isomorphic contact graphs, and the same (ambient isotopic) links obtained from their middle graphs. This opens new possibilities for studying graphical recombination patterns via the Tutte polynomial.

6 Recursive formulae for the Tutte polynomials

In this section we derive recursive formulae for the Tutte polynomias for the simplest non-hollow polyominoes (polyiamonds, polyhexes) and graphs corresponding to a few protein structures.

The simplest among graphs obtained from non-hollow polyominoes (polyiamonds, polyhexes) are those with contact graphs equal to L_n, for some n, where L_n denotes a rectilinear graph with n vertices.

For such polyiamond graphs we have the following recursion (Fig. 7):

Theorem 1 *The Tutte polynomial of a non-hollow n-polyiamond graph P_n with a contact graph L_n ($n \geq 2$) is given by*

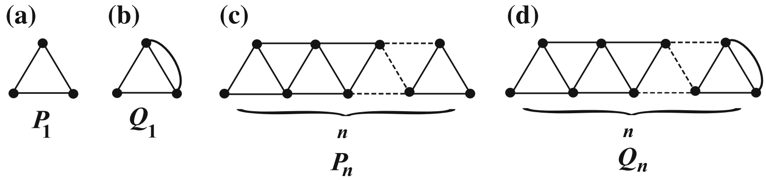

Fig. 7 Polyiamond graphs **a** P_1, **b** Q_1, **c** P_n, and **d** Q_n

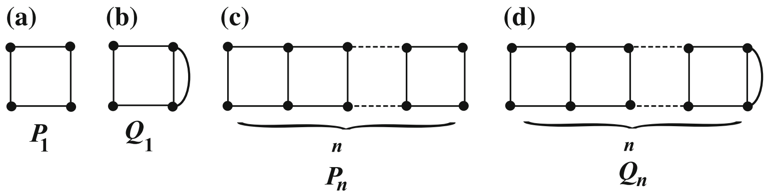

Fig. 8 Polyomino graphs **a** P_1, **b** Q_1, **c** P_n, and **d** Q_n

$$T(P_n) = xT(P_{n-1}) + T(Q_{n-1})$$
$$T(Q_n) = T(P_n) + yT(Q_{n-1})$$

with the generators

$$T(P_1) = x + x^2 + y$$
$$T(Q_1) = x + x^2 + y + xy + y^2.$$

Examples of simple polyomino graphs are shown on Fig. 8.

Theorem 2 *The Tutte polynomial of a non-hollow n-polyomino graph P_n with a contact graph L_n ($n \geq 2$) is given by*

$$T(P_n) = (x^2 + x)T(P_{n-1}) + T(Q_{n-1})$$
$$T(Q_n) = T(P_n) + xyT(P_{n-1}) + yT(Q_{n-1})$$

with the generators

$$T(P_1) = x + x^2 + x^3 + y$$
$$T(Q_1) = x + x^2 + x^3 + y + xy + x^2y + y^2.$$

For the simplest polyhex graphs (Fig. 9) we have the following recursion:

Theorem 3 *The Tutte polynomial of a non-hollow n-polyhex graph P_n with a contact graph L_n ($n \geq 2$) is given by*

$$T(P_n) = (x^4 + x^3 + x^2 + x)T(P_{n-1}) + T(Q_{n-1})$$
$$T(Q_n) = T(P_n) + y(x^3 + x^2 + x)T(P_{n-1}) + yT(Q_{n-1})$$

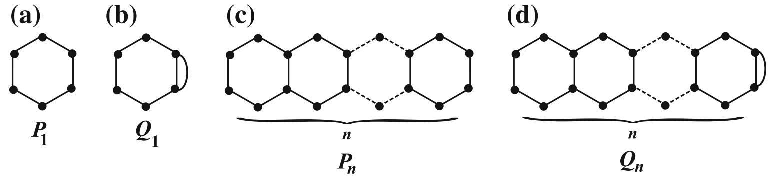

Fig. 9 Polyhex graphs **a** P_1, **b** Q_1, **c** P_n, and **d** Q_n

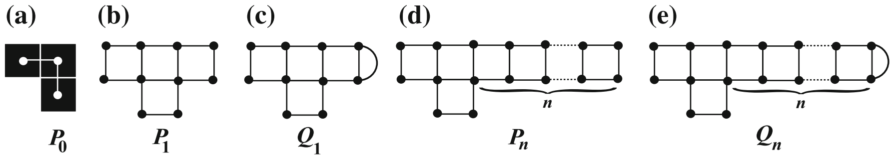

Fig. 10 Polyomino graphs **a** P_0, **b** P_1, **c** Q_1, **d** P_n, **e** Q_n

with the generators

$$T(P_1) = x + x^2 + x^3 + x^4 + x^5 + y$$
$$T(Q_1) = x + x^2 + x^3 + x^4 + x^5 + y + xy + x^2y + x^3y + x^4y + y^2.$$

Preceding results can be extended to more complex, non-hollow polyiamond, polyomino, or polyhex graphs by introducing different generators P_1 and Q_1.

For example, given two generating graphs P_1 and Q_1 (Fig. 10b, c), we can compute the Tutte polynomial of the polyomino graph P_n, obtained from P_1 by continuing the chain of squares (Fig. 10d), based on Theorem 2, and we obtain the following results:

$$\begin{aligned} T(P_1) = {} & x + 4x^2 + 10x^3 + 16x^4 + 19x^5 + 16x^6 + 10x^7 + 4x^8 + x^9 + y + 6xy \\ & +15x^2y + 22x^3y + 21x^4y + 12x^5y + 4x^6y + 3y^2 + 9xy^2 \\ & +12x^2y^2 + 9x^3y^2 + 3x^4y^2 + 3y^3 + 3xy^3 + 3x^2y^3 + y^4 \\ T(Q_1) = {} & x + 4x^2 + 10x^3 + 16x^4 + 19x^5 + 16x^6 + 10x^7 + 4x^8 + x^9 + y \\ & +7xy + 19x^2y + 31x^3y + 34x^4y + 25x^5y + 13x^6y + 4x^7y + x^8y + 4y^2 \\ & +15xy^2 + 26x^2y^2 + 27x^3y^2 + 17x^4y^2 + 6x^5y^2 + x^6y^2 + 6y^3 + 12xy^3 \\ & +13x^2y^3 + 6x^3y^3 + x^4y^3 + 4y^4 + 3xy^4 + 2x^2y^4 + y^5. \end{aligned}$$

Since all non-hollow polyominoes with isomorphic contact graphs have the same alternating links corresponding to their graphs and the same Tutte polynomial, we can apply this result to a large classes of polyominoes, polyiamonds, and polyhexes.

Jones polynomials of these links are obtained from their Tutte polynomials by replacements: $x \to -x$ and $y \to -\frac{1}{x}$.

"Portraits of families" are plots of zeros of Jones polynomials corresponding to different families of *KL*s associated to non-hollow *n*-minoes. All results are obtained in *Mathematica 7.0* (see Figs. 11, 12, 13 and 17, 18, 19).

Motivated by the work of F. Emmert-Streib [22], where he gives recursive formulae for the Tutte polynomial of antiparallel [22, Theorem 4.1], and the helix sheet

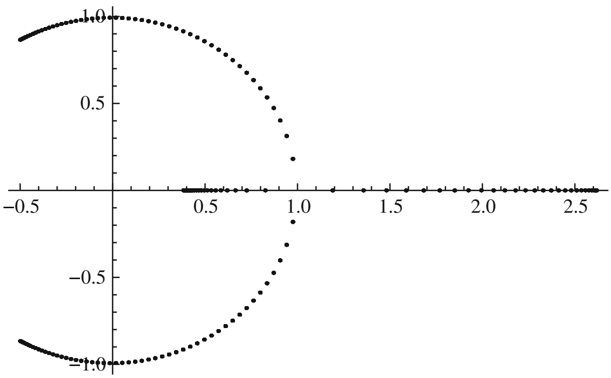

Fig. 11 "Portrait of family" for polyiamonds, $n \leq 120$

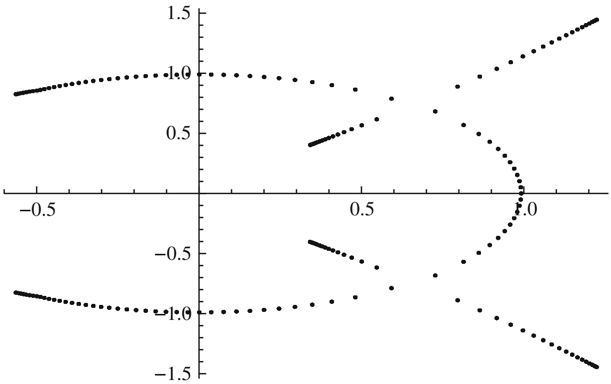

Fig. 12 "Portrait of family" for polyiominoes, $n \leq 100$

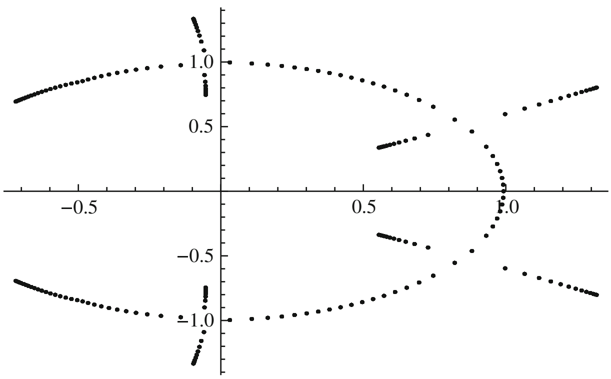

Fig. 13 "Portrait of family" for polyhexes, $n \leq 80$

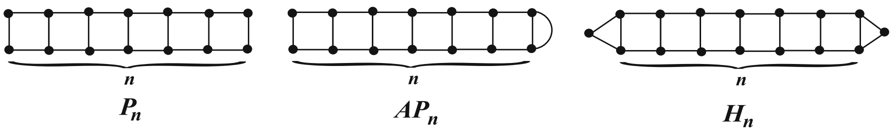

Fig. 14 Polyomino graphs P_n, AP_n, and H_n

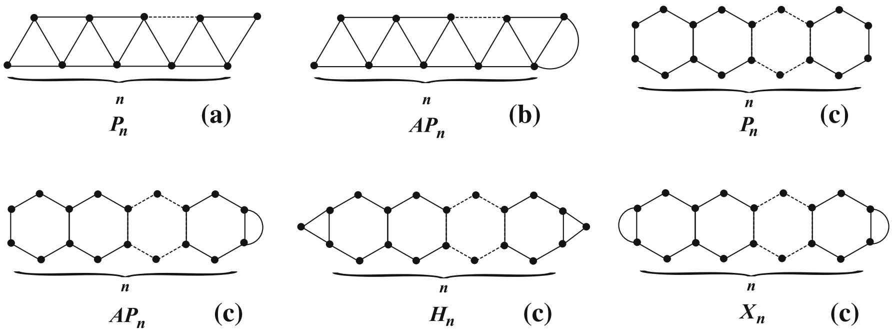

Fig. 15 **a** Antiparallel sheet for polyiamonds; **b** antiparallel sheet for polyhexes; **c** polyhex graphs H_n and X_n

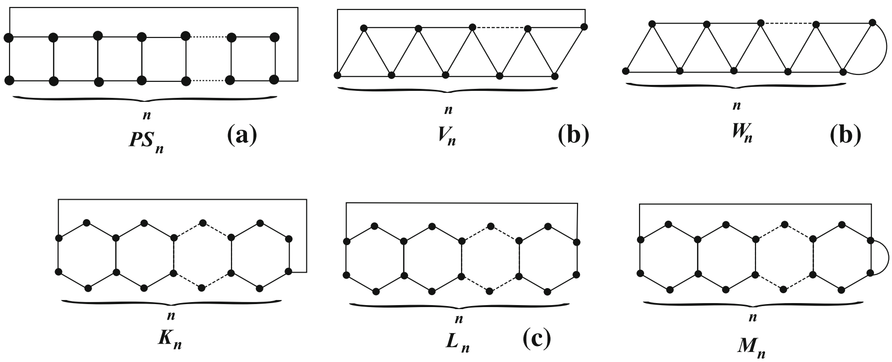

Fig. 16 **a** Parallel sheet PS_n for polyominoes [22, Theorem 4.5]; **b** parallel sheet V_n for polyiamonds and the graph W_n; **c** parallel sheet K_n for polyhexes and the graphs L_n and M_n

[22, Theorem 4.3], we derive similar formulas with recursion depending on different graphs. If we denote antiparallel graph by AP_n, and helix by H_n (Fig. 14), we can express recursion in terms of the Tutte polynomials $T(P_n)$ corresponding to n-minoes (Theorem 2):

$$T(AP_n) = T(P_n) + xyT(P_{n-1}) + yT(AP_{n-1})$$

with the generator

$$T(AP_1) = x + x^2 + x^3 + y + xy + x^2y + y^2$$

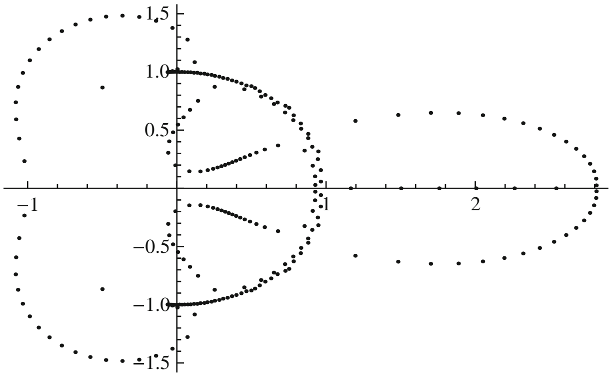

Fig. 17 “Portrait of family” for polyiamond graphs V_n for $n \leq 120$

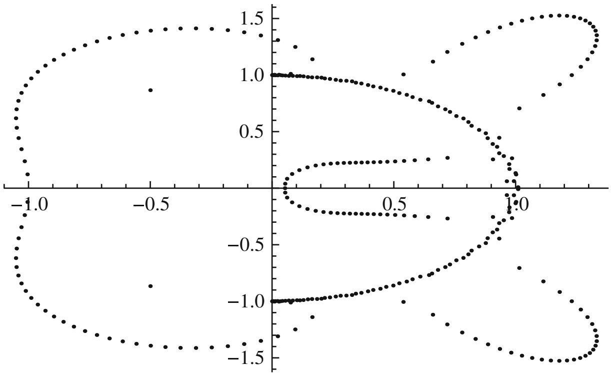

Fig. 18 “Portrait of family” for polyomino graphs PS_n for $n \leq 100$

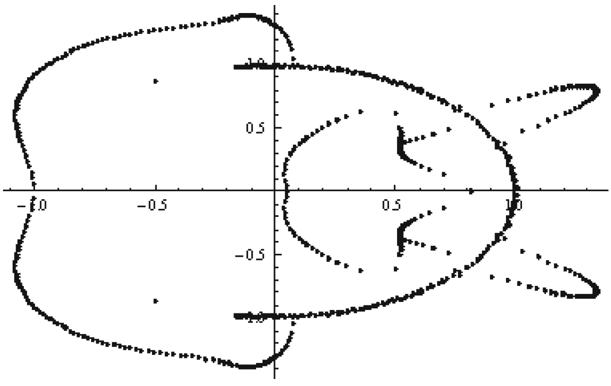

Fig. 19 “Portrait of family” for polyhex graphs K_n for $n \leq 120$

for antiparallel sheet, and

$$T(H_n) = xAP_n + x^2P_n + xP_n + AP_n + yH_{n-1}$$

with the generator

$$T(H_1) = x + 3x^2 + 4x^3 + 3x^4 + x^5 + y + 4xy + 5x^2y + 2x^3y + 2y^2 + 3xy^2 + y^3$$

for helix.

The same method can be applied to antiparallel sheet derived from polyiamonds[2] (Fig. 15a), and antiparallel and helix sheets (Fig. 15b, c) derived from polyxexes, based on the recursions from Theorem 1 and Theorem 3, respectively.

Theorem 4 *For antiparallel sheet derived from polyiamonds (Fig. 15a) we have the following recursive formula for the Tutte polynomial*

$$T(AP_n) = T(P_n) + yT(AP_{n-1})$$

with the generator

$$T(AP_1) = x + x^2 + y + xy + y^2.$$

Theorem 5 *The recursive formula for the Tutte polynomial of antiparallel sheet derived from polyhexes (Fig. 11b)*

$$T(AP_n) = T(P_n) + y(x^3 + x^2 + x)T(P_{n-1}) + yT(AP_{n-1})$$

is generated by

$$T(AP_1) = x + x^2 + x^3 + x^4 + x^5 + y + xy + x^2y + x^3y + x^4y + y^2.$$

For for the Tutte polynomial of helix graph derived from polyhexes (Fig. 11c) we obtain the following formula:

$$\begin{aligned}
T(H_n) &= x^2T(P_n) + xT(AP_n) + xT(P_n) + T(AP_n) + x^4yT(P_{n-1}) + x^3yT(P_{n-1}) \\
&\quad + x^2yT(P_{n-1}) + x^3yT(AP_{n-1}) + x^2yT(AP_{n-1}) \\
&\quad + 2xyT(AP_{n-1}) + yT(X_{n-1}) \\
T(X_n) &= T(AP_n) + x^3yT(AP_{n-1}) + x^2yT(AP_{n-1}) + xyT(AP_{n-1}) + xyT(X_{n-1})
\end{aligned}$$

generated by

$$\begin{aligned}
T(X_1) &= x + x^2 + x^3 + x^4 + x^5 + y + 2xy + 2x^2y + 2x^3y + 2x^4y + 2y^2 + xy^2 \\
&\quad + x^2y^2 + x^3y^2 + y^3,
\end{aligned}$$

[2] Helix sheet derived from n-polyiamond is $(n + 2)$-polyiamond.

The obtained recursions hold for all antiparallel and helix sheets with isomorphic contact graphs. Moreover, by changing generators we can obtain Tutte polynomials corresponding to more complex antiparallel and helix graphs.

The most interesting "portrait of family" is the plot of zeros (Fig. 18) of the Jones polynomials of KLs corresponding to polyomino parallel sheet PS_n. Figure 16a shows the plot computed for $n \leq 100$, based on the recursion for Tutte polynomials derived by F. Emmert-Streib [22, Theorem 4.5].

Similar results are obtained for parallel sheet corresponding to polyiamonds.

Theorem 6 *For the Tutte polynomial of the family of graphs V_n (Fig. 16b) we have the following recursion:*

$$\begin{aligned} W_n &= V_n + yW_{n-1} \\ V_n &= xP_{n-1} + V_{n-1} + xV_{n-2} + W_{n-2} + yW_{n-2} \end{aligned}$$

with the generators

$$\begin{aligned} W_2 &= 2x + 3x^2 + x^3 + 2y + 5xy + x^2y + 4y^2 + 2xy^2 + 3y^3 + y^4 \\ V_3 &= 3x + 6x^2 + 4x^3 + x^4 + 3y + 9xy + 4x^2y + 6y^2 + 4xy^2 + 4y^3 + y^4 \\ V_4 &= 5x + 12x^2 + 11x^3 + 5x^4 + x^5 + 5y + 19xy + 16x^2y + 4x^3y \\ &\quad + 12y^2 + 16xy^2 + 3x^2y^2 + 11y^3 + 4xy^3 + 5y^4 + y^5 \end{aligned}$$

where W_n are graphs from Fig. 16b.

Theorem 7 *For the Tutte polynomial of the family of graphs K_n (Fig. 16c) we have the following recursion:*

$$\begin{aligned} T(L_n) &= (x^4 + 2x^3 + 2x^2 + x)T(P_{n-1}) + (x^2 + x)T(L_{n-1}) \\ &\quad + (x+1)T(K_{n-1}) + T(M_{n-1}) \\ T(M_n) &= T(L_n) + (x^3y + 2x^2y + xy)T(P_{n-1}) + (xy + y)T(K_{n-1}) \\ &\quad + yT(M_{n-1}) + xyT(L_{n-1}) \\ T(K_n) &= (x^4 + 2x^3 + 2x^2 + x)T(P_{n-1}) + (x^2 + x + 1)T(K_{n-1}) \\ &\quad + xT(L_{n-1}) + T(M_{n-1}) \\ T(L_2) &= 2x + 7x^2 + 13x^3 + 16x^4 + 14x^5 + 10x^6 + 6x^7 + 3x^8 + x^9 + 2y + 8xy \\ &\quad + 12x^2y + 10x^3y + 4x^4y + x^5y + 3y^2 + 4xy^2 + 2x^2y^2 + y^3 \end{aligned}$$

with the generators

$$\begin{aligned} T(K_2) &= 2x + 7x^2 + 13x^3 + 16x^4 + 15x^5 + 10x^6 + 6x^7 + 3x^8 + x^9 + 2y \\ &\quad + 8xy + 12x^2y + 10x^3y + 5x^4y + 3y^2 + 4xy^2 + 2x^2y^2 + y^3 \\ T(M_2) &= 2x + 7x^2 + 13x^3 + 16x^4 + 14x^5 + 10x^6 + 6x^7 + 3x^8 + x^9 + 2y + 10xy \\ &\quad + 19x^2y + 22x^3y + 17x^4y + 11x^5y + 6x^6y + 3x^7y + x^8y + 5y^2 \\ &\quad + 12xy^2 + 13x^2y^2 + 7x^3y^2 + 2x^4y^2 + 4y^3 + 4xy^3 + x^2y^3 + y^4 \end{aligned}$$

where K_n, L_n, and M_n are graphs from Fig. 16c.

"Portraits of families", the plots of zeros of the Jones polynomials corresponding to polyiamond and polyhex parallel sheet, computed by using the recursion for Tutte polynomials from the Theorems 6 and 7, are given in Figs. 17 and 19, respectively.

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References

1. J. Conway, in *An enumeration of knots and links and some of their related properties*, ed. by J. Leech Computational Problems in Abstract Algebra, Proceedings of Conferece in Oxford 1967. (Pergamon Press, New York, 1970), pp. 329–358
2. D. Rolfsen, *Knots and Links*. (Publish & Perish Inc., Berkeley, 1976; American Mathematical Society, AMS Chelsea Publishing, 2003)
3. A. Caudron, *Classification des nœuds et des enlancements*. (Public. Math. d'Orsay 82. Univ. Paris Sud, Dept. Math., Orsay, 1982)
4. S.V. Jablan, R. Sazdanović, *LinKnot- Knot Theory by Computer*. (World Scientific, New Jersey, London, Singapore, 2007; http://math.ict.edu.rs/)
5. B. Thompson, D.J. Pearce, http://homepages.mcs.vuw.ac.nz/~djp/files/TPH-SIENZ07.pdf, (2008)
6. J. Ellis-Monaghan, C. Merino, Graph Polynomials and their Applications I: The Tutte Polynomial, arXiv:0803.3079v1 [math] (2008)
7. M. Thistlethwaite, A spanning tree expansion of the Jones polynomial. Topology **26**, 297–309 (1987)
8. L.H. Kauffman, A Tutte polynomial for signed graphs. Discrete Appl. Math. **25**, 105–127 (1989)
9. B. Bollobás, *Modern Graph Theory* (Springer, Berlin, 1998)
10. S.-C. Chang, R. Shrock, Zeroes of Jones Polynomials for Families of Knots and Links, arXiv:math-ph/0103043v2 (2001)
11. F. Jaeger, Tutte polynomials and link polynomials. Proc. Am. Math. Soc. **103**, 647–654 (1988)
12. X.-S. Lin, Zeroes of Jones polynomial, http://math.ucr.edu/~xl/abs-jk.pdf
13. F.Y. Wu, J. Wang, Zeroes of Jones polynomial. Physica A **269**, 483–494 (2001)
14. X. Jin, F. Zhang, Zeroes of the Jones polynomials for families of pretzel links. Physica A **328**, 391–408 (2003)
15. A. Champanerkar, I. Kofman, On the Mahler measure of Jones polynomials under twisting. Algebraic Geometr. Topol. **5**, 1–22 (2005)
16. S.V. Jablan, Lj. Radović, R. Sazdanović, Tutte and Jones Polynomial of Link Families, arXiv:1004.4302v1 [mathGT] (2010)
17. D.J.A. Welsh, C. Merino, The Potts model and the Tutte polynomial. J. Math. Phys. **41**(3), 1127–1152 (2000)
18. G. Chartrand, *Introductory Graph Theory* (Dover, New York, 1985)
19. F. Harary, *Graph Theory* (Addison-Wesley, Reading, MA, 1994)
20. B. Kurpita, K. Murasugi, Knots and graphs. Chaos Soliton. Fract. **9**(4/5), 623–643 (1998)
21. S.-J. Chen, K.A. Dill, Symmetries in proteins: a knot-theory approach. J. Chem. Phys. **104**, 5964–5973 (1996)
22. F. Emmert-Streib, Algorithmic computation of knot polynomials of secondary structure elements of proteins. J. Comput. Biol. **13**(8), 1503–1512 (2006)
23. I. Gutman, S.J. Cyvin, *Introduction to the Theory of Benzenoid Hydrocarbons* (Springer, Berlin, 1989)
24. J. Rangel-Mondragón, Polyominoes and related families. Math. J. **9**(3), 609–640 (2005)
25. C. Liang, Y. Jiang, The chirality of ground DNA knots and links. J. Theor. Biol. **158**, 231–243 (1982)